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STONE'S TOPOLOGY FOR PSEUDOCOMPLEMENTED AND BICOMPLEMENTED LATTICES(1)

BY

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ABSTRACT. In an earlier paper the author has proved the existence of prime ideals and prime dual ideals in a pseudocomplemented lattice (not necessarily distributive). The present paper is devoted to a study of Stone's topology on the set of prime dual ideals of a pseudocomplemented and a bicomplemented lattice.

If $\hat{L}$ is the quotient lattice arising out of the congruence relation defined by $a \equiv b \Leftrightarrow a^* = b^*$ in a pseudocomplemented lattice L, it is proved that Stone's space of prime dual ideals of $\hat{L}$ is homeomorphic to the subspace of maximal dual ideals of L.

Stone [9] has introduced a topology for the set of all prime ideals of a distributive lattice. Balachandran [3] has made an extensive study of Stone's topology of the distributive lattice and has obtained results supplementing those of Stone. The purpose of this paper is to extend some of the results of Stone and Balachandran to pseudocomplemented and bicomplemented lattices. (A lattice closed for pseudocomplements as well as quasicomplements is called a bicomplemented lattice.)

In the first section we collect some known results which are used in subsequent sections. §§2 and 3 deal with Stone's topology on the set of prime dual ideals of a pseudocomplemented lattice and a bicomplemented lattice respectively. The concluding section is devoted to a study of ideals and dual ideals of the quotient lattice of a pseudocomplemented lattice with respect to a special congruence relation. It is proved that Stone's space of prime dual ideals of the quotient lattice is homeomorphic to the subspace of maximal dual ideals of the given lattice.

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1. **Preliminaries.** For the lattice-theoretic and topological concepts which have now become commonplace the reader is referred to [4], [6] and [10]. We shall recall some concepts which are not quite well known.

Let a be an element of a semilattice S with 0. Then an element a^* of S is called a *pseudocomplement* of a if (i) $aa^* = 0$ and (ii) $ab = 0 \Rightarrow b \leqq a^*$. An element $\bar{a}$ of a dual semilattice S with 1 is called a *quasicomplement* of an element a of S if (i) $a + \bar{a} = 1$ and (ii) $a + b = 1 \Rightarrow b \geqq \bar{a}$. A semilattice in which every element has a pseudocomplement is said to be *pseudocomplemented.* Similarly we define a *quasicomplemented* dual semilattice. An element a of a pseudocomplemented semilattice is said to be *normal* (*dense*) if $a = a^{**}$ ($a^* = 0$). A subset A of a partially ordered set P is called a *semi-ideal* if $a \in A$, $b \leqq a \Rightarrow b \in A$. A semi-ideal A of P is called an *ideal* if the lattice sum of any finite number of elements of A, whenever it exists, belongs to A. *Dual semi-ideal* and *dual ideal* are defined in a dual fashion. Obviously, in a lattice the above definitions coincide with the usual definitions. An element a of a pseudocomplemented lattice is said to be *simple* if $a + a^* = 1$. A pseudocomplemented lattice in which every normal element is simple is called an *S-lattice*. A distributive S-lattice is called a *Stone lattice.*

The results summarized in the form of Theorem I below are due to Frink [5].

Theorem I. *In any pseudocomplemented semilattice S, the following results hold*: (i) $a \leqq a^{**}$, (ii) $a^{***} = a^*$, (iii) $a \leqq b \Rightarrow a^* \geqq b^*$, (iv) $(ab)^* = (a^{**}b^{**})^*$, (v) $(ab)^{**} = a^{**}b^{**}$, (vi) *S has the greatest element* 1 *and* $0^* = 1$, (vii) *a is normal if and only if $a = b^*$ for some b,* (viii) *the set N of normal elements of S forms a Boolean algebra; the lattice sum of any two elements a, b of N is $(a^*b^*)^*$ and their lattice product is the same as that in L,* (ix) *if L is a lattice,* $(a + b)^* = a^*b^*$ *and* $a^* + b^* \leqq (ab)^*$.

Let X be a topological space. A point p of X is said to be T_1 (*anti-*T_1) if $q \notin$ closure of p ($p \notin$ closure of q) for any point q of X other than p. X is said to be Π_0 if every non-null open subset of X contains a non-null closed subset. Let A and B be two disjoint subsets of X. Then we say A is *weakly separable* from B if there exists an open subset of X containing A and disjoint with B. It is easily seen that A is weakly separable from B if and only if $A \cap \mathrm{cl}\, B = \phi$ (ϕ denotes the empty set).

We need the following lemmas in the sequel.

Lemma I [13]. *Any proper dual ideal of a partially ordered set with* 0 *is contained in a maximal dual ideal.*

Lemma II [14]. *Any maximal dual ideal of a pseudocomplemented lattice is prime.*

Lemma III [13]. *The set* D *of all dense elements of a partially ordered set* P *with* 0,1 *forms a dual ideal and* D *is the product of all the maximal dual ideals of* P.

Lemma IV [14]. *Any prime ideal of a pseudocomplemented semilattice contains a minimal prime ideal.*

Lemma V [11]. *If in a modular lattice, the lattice sum and lattice product of two ideals are principal, then the ideals themselves are principal.*

Lemma VI [14]. *A prime ideal of a pseudocomplemented semilattice* S *is minimal prime if and only if it contains precisely one of* x, x^* *for every* x *in* S.

For convenience we shall prove the 'only if' part of Lemma VI here.

First we observe that any maximal dual ideal M of S contains precisely one of x, x^* for every x in S. (Since $xx^* = 0$, it is clear that M contains at most one of x, x^*. If $x \notin M$, then $M \vee [x) = S$. Hence $yx = 0$ for some y in M; $y \leqq x^*$ and so $x^* \in M$.) Now we shall prove that if A is a minimal prime ideal of S, then the set complement cA of A is a maximal dual ideal. Clearly cA is a dual ideal. By Lemma I, cA is contained in a maximal dual ideal M of S. $A \supseteqq cM$ and clearly cM is a semi-ideal. Let $x_1, x_2, \cdots, x_n \in cM$ and suppose $x_1 + x_2 + \cdots + x_n$ exists. Since M is a maximal dual ideal, $x_1^*, x_2^*, \cdots, x_n^* \in M$ and so $x_1^* x_2^* \cdots x_n^* \in M$. That is $(x_1 + x_2 + \cdots + x_n)^* \in M$. Hence $x_1 + x_2 + \cdots + x_n \in cM$. Thus cM is an ideal. It is easily seen that cM is prime. By the minimality of A, it follows that $A = cM$. Hence $cA = M$, so that cA is a maximal dual ideal. From the above it follows that every minimal prime ideal of S contains precisely one of x, x^* for every x in S.

Lemma VII [11]. *If in a pseudocomplemented modular lattice* L, $D = [1)$, *then* L *is a Boolean algebra.*

Lemma VIII. *The first three of the following statements concerning a pseudocomplemented lattice* L *are equivalent and each of these is implied by the fourth.*

(i) *Every prime ideal is minimal prime.*

(ii) *Every prime dual ideal is minimal prime.*

(iii) *Every prime dual ideal is maximal.*

(iv) $D = [1)$.

Proof. Suppose A is a prime dual ideal which is not minimal prime. Then there exists a prime dual ideal B such that $B \subsetneqq A$; cA, cB are prime ideals and $cA \subsetneqq cB$ (cA, cB are the set complements of A, B respectively). It follows that cB is not minimal prime. Thus, (i) $\Rightarrow$ (ii).

Let C be a prime dual ideal which is not maximal. Then by Lemma I, there exists a maximal dual ideal M such that $M \supsetneqq C$. By Lemma II, M is prime. Thus M is a prime dual ideal which is not minimal prime. Hence (ii) $\Rightarrow$ (iii).

Let A be a prime ideal which is not minimal prime. Then by Lemma IV, there is a minimal prime ideal B such that $B \subsetneqq A$. Clearly cA and cB are proper prime dual ideals and $cB \supsetneqq cA$. Thus cA is a prime dual ideal which is not maximal. Hence (iii) $\Rightarrow$ (i).

Suppose (iv) holds. Then for every $a \in L$, $a + a^* = 1$ and so every prime ideal contains precisely one of a, a^*. Hence by Lemma VI, every prime ideal is minimal prime. Thus (iv) $\Rightarrow$ (i).

2. **Stone's topology of the pseudocomplemented lattice.** In this section we extend some results of Stone [9] and Balachandran [3] on Stone's topology of the distributive lattice to pseudocomplemented lattices.

Throughout this section L denotes a pseudocomplemented lattice and $\mathfrak{D}$ the lattice of dual ideals of L. We denote by $\mathfrak{P}$ the set of all prime dual ideals of L. For any dual ideal A of L, $F(A)$ denotes the set of all prime dual ideals of L containing A and $F'(A)$ the set complement of $F(A)$ in $\mathfrak{P}$.

The lattice sum and lattice product of two elements a, b of L are denoted by $a + b$ and ab respectively. The pseudocomplement of a is denoted by a^*. We shall denote the lattice sums in the lattice of all ideals as well as the lattice of all dual ideals of L by $\vee$. The lattice products in these lattices coincide with the corresponding set intersections. Set inclusion, set union and set intersection are denoted by $\subseteqq$, $\cup$ and $\cap$ respectively.

Theorems 1, 2 and 3 below, enunciated without proofs, are analogues of the corresponding results in §6 of [12] and the proofs of these are similar to those of the corresponding results of [12] with obvious modifications.

Theorem 1. (i) $F(\bigvee_{i\in I} A_i) = \bigcap_{i \in I} F(A_i)$, (ii) $F(A_1 \cap A_2 \cap \cdots \cap A_n) = F(A_1) \cup F(A_2) \cup \cdots \cup F(A_n)$, (iii) $F(L) = \phi$, (iv) $F([1)) = \mathfrak{P}$.

(*Here the* A_i *are dual ideals of* L.)

Corollary. (i) $F'(\bigvee_{i\in I} A_i) = \bigcup_{i\in I} F'(A_i)$, (ii) $F'(A_1 \cap A_2 \cap \cdots \cap A_n) = F'(A_1) \cap F'(A_2) \cap \cdots \cap F'(A_n)$, (iii) $F'(L) = \mathfrak{P}$, (iv) $F'([1)) = \phi$.

Theorem 1 shows that F defines a closure operation in $\mathfrak{P}$ thereby giving rise to a topology T on $\mathfrak{P}$.

Theorem 2. *The lattice of all open (closed) sets of* $(\mathfrak{P}, T)$ *is homomorphic (dually homomorphic) to* $\mathfrak{D}$ *and the mapping* $A \to F'(A)$ $(A \to F(A))$ *takes arbitrary lattice sums to corresponding set unions (set intersections).*

Theorem 3. *If* X *is any subset of* $\mathfrak{P}$, $\mathrm{cl}\, X = F(X_0)$, X_0 *being the product of all the members of* X.

Theorem 4. *$(\mathcal{P}, T)$ is T_0.*

Theorem 4 follows immediately from Theorem 3.

Theorem 5. *$(\mathcal{P}, T)$ is compact.*

Proof. Let $\mathcal{P} = \bigcup_{i\in I} F'(A_i)$. Then by the corollary under Theorem 1, $F'(L) = F'(\bigvee_{i\in I} A_i)$. Therefore $\bigvee_{i\in I} A_i = L$ (for, otherwise by Lemmas I and II, there would exist a prime dual ideal containing $\bigvee_{i\in I} A_i$ which is a contradiction to the fact that $F'(\bigvee_{i\in I} A_i) = \mathcal{P}$). Hence there exist a finite number of elements a_{i_1}, $a_{i_2}, \cdots, a_{i_n}$ $(a_{i_j} \in A_{i_j})$ such that $0 = a_{i_1} a_{i_2} \cdots a_{i_n}$. It follows that $\bigvee_{i\in I} A_i = [a_{i_1}) \vee [a_{i_2}) \vee \cdots \vee [a_{i_n}) \subseteqq A_{i_1} \vee A_{i_2} \vee \cdots \vee A_{i_n}$. Consequently

$$\mathcal{P} \subseteqq F'(A_{i_1} \vee A_{i_2} \vee \cdots \vee A_{i_n}) = F'(A_{i_1}) \cup F'(A_{i_2}) \cup \cdots \cup F'(A_{i_n}).$$

Hence the result.

Theorem 6. *The closure of the set of T_1 points of $(\mathcal{P}, T)$ is $F(D)$ where D is the dual ideal of dense elements of L.*

Proof. By Lemma II, every maximal dual ideal of L is an element of $\mathcal{P}$. It is easily seen that the maximal dual ideals are precisely the T_1 points of $(\mathcal{P}, T)$. Also, by Lemma III, D is the product of all the maximal dual ideals of L. Hence the result follows by Theorem 3.

Theorem 7. *$(\mathcal{P}, T)$ is Π_0 if $D = [1)$.*

Proof. Let $g = F'(A)$ be a non-null open subset of $(\mathcal{P}, T)$ and let $D = [1)$. Then $A \neq [1)$ and so there exists a maximal dual ideal M of L not containing A. (For, otherwise by Lemma III, $A \subseteqq D = [1)$.) Also, by Lemma II, it follows that $M \in \mathcal{P}$. Thus g contains the closed set consisting of the single point M. Hence $(\mathcal{P}, T)$ is Π_0.

Theorem 8. *$(\mathcal{P}, T)$ is normal if and only if L is an S-lattice.*

Proof. Suppose L is an S-lattice. Let $F_1 = F(A)$, $F_2 = F(B)$ be any two disjoint closed subsets of $\mathcal{P}$. Then, by Theorem 1, $F(A \vee B) = \phi$. Hence $A \vee B = L$. (For, otherwise by Lemmas I and II there would exist a prime dual ideal containing $A \vee B$, contradicting the fact that $F(A \vee B) = \phi$.) Hence there exist $a \in A$ and $b \in B$ such that $ab = 0$; $a^* \geqq b$ and so $a^* \in B$. Set $F_3 = F([a^{**}))$ and $F_4 = F([a^*))$. Then

$$F_1 \cap F_4 = F(A) \cap F([a^*)) \subseteqq F([a)) \cap F([a^*))$$

$$= F([a) \vee [a^*)) = F([aa^*)) = F([0)) = F(L) = \phi;$$

$$F_2 \cap F_3 = F(B) \cap F([a^{**})) \subseteqq F([a^*)) \cap F([a^{**}))$$

$$= F([a^*) \vee [a^{**})) = F([a^* a^{**})) = F([0)) = F(L) = \phi.$$

Since $a^{**} \geqq a$, $a^{**} \in A$ and so $F_3 \supseteqq F_1$. As $a^* \in B$, $F_4 \supseteqq F_2$. Also $F_3 \cup F_4 = F([a^{**})) \cup F([a^*)) = F([a^{**}) \cap [a^*)) = F([a^{**} + a^*)) = F([1)) = \mathcal{P}$. From the above it follows that $(\mathcal{P}, T)$ is normal.

Conversely, suppose $(\mathcal{P}, T)$ is normal and $x \in L$. Set $F_1 = F([x))$ and $F_2 = F([x^*))$. Then clearly $F_1 \cap F_2 = \phi$. Hence, as $(\mathcal{P}, T)$ is normal, there exist closed subsets $F_3 = F(A)$, $F_4 = F(B)$ containing F_1, F_2 respectively such that

(1) $$F_1 \cap F_4 = \phi,$$

(2) $$F_2 \cap F_3 = \phi,$$

(3) $$F_3 \cup F_4 = \mathcal{P}.$$

From (1) we have $[x) \vee B = L$. Hence $xb = 0$ for some $b \in B$. Clearly

(4) $$x^* \geqq b.$$

Similarly, using (2), we can prove that, for some $a \in A$,

(5) $$x^{**} \geqq a.$$

From (3) we have $A \cap B = [1)$. Hence $[a) \cap [b) = [1)$ and so $a + b = 1$. From (4) and (5) it follows that $x^* + x^{**} = 1$. Thus L is an S-lattice.

Let us denote the set of all maximal dual ideals and the set of all dense maximal dual ideals of L by $\mathfrak{M}$ and $\mathfrak{M}_1$ respectively. By Lemma II, $\mathfrak{M} \subseteqq \mathcal{P}$.

Clearly every point of $\mathfrak{M}$ is T_1. Hence we have the following

Theorem 9. *The subspaces* $(\mathfrak{M}, T)$ *and* $(\mathfrak{M}_1, T)$ *are* T_1.

Theorem 10. *The subspace* $(\mathfrak{M}, T)$ *is the smallest of the subspaces* X *of* $(\mathcal{P}, T)$ *such that* X *is not weakly separable from any point outside it.*

Proof. Let $A \in \mathcal{P}$ and $A \notin \mathfrak{M}$. By Lemma I, there exists $M \in \mathfrak{M}$ such that $M \supseteqq A$. Thus $M \in \mathfrak{M} \cap F(A)$ and so $\mathfrak{M} \cap F(A) \neq \phi$; i.e. $\mathfrak{M} \cap \mathrm{cl}\,A \neq \phi$. Hence $\mathfrak{M}$ is not weakly separable from any point outside it.

Now suppose X is a subspace of $(\mathcal{P}, T)$ such that $\mathfrak{M} \not\subseteqq X$. Then there exists $M \in \mathfrak{M}$ such that $M \notin X$. Clearly $\{M\}$ is closed in $(\mathcal{P}, T)$ and so $X \cap \mathrm{cl}\{M\} = \phi$. Hence X is weakly separable from M.

Theorem 11. $\mathfrak{M}$ *is closed in* $(\mathcal{P}, T)$.

Proof. By Theorem 3, $\mathrm{cl}\,\mathfrak{M} = F(D)$. Let $A \in \mathrm{cl}\,\mathfrak{M}$. We shall prove that $A \in \mathfrak{M}$. Let $x \in L$ and $x \notin A$. Since $(x + x^*)^* = 0$, $x + x^* \in D \subseteqq A$. Hence, as A is prime and $x \notin A$, $x^* \in A$. Consequently, $0 = xx^* \in A \vee [x)$ and so $A \vee [x) = L$. Hence $A \in \mathfrak{M}$. It follows that $\mathrm{cl}\,\mathfrak{M} = \mathfrak{M}$. Hence the result.

Corollary. *The subspace* $(\mathfrak{M}, T)$ *is compact.*

Theorem 12. *If* X *is any subset of* $\mathcal{P}$ *containing* $\mathfrak{M}$, *then* (X, T) *is compact.*

Proof. Let $X \subseteqq \bigcup_{i\in I} F'(A_i)$. Then $X \subseteqq F'(\bigvee_{i\in I} A_i)$ and so no member of X contains $\bigvee_{i\in I} A_i$. In particular no member of $\mathfrak{M}$ contains $\bigvee_{i\in I} A_i$. By Lemma I, it follows that $\bigvee_{i\in I} A_i = L$. Hence $0 = a_{i_1} a_{i_2} \cdots a_{i_n}$ (for some $a_{i_j} \in A_{i_j}$) so that $L = [a_{i_2} a_{i_2} \cdots a_{i_n}) = [a_{i_1}) \vee [a_{i_2}) \vee \cdots \vee [a_{i_n}) \subseteqq A_{i_1} \vee A_{i_2} \vee \cdots \vee A_{i_n}$. It follows that

$$X \subseteqq F'(A_{i_1} \vee A_{i_2} \vee \cdots \vee A_{i_n}) = F'(A_{i_1}) \cup F'(A_{i_2}) \cup \cdots \cup F'(A_{i_n}).$$

Hence the result.

Theorem 13. *The subspace* $(\mathfrak{M}_1, T)$ *is compact.*

Proof. Let D_1 be the product of all the members of $\mathfrak{M}_1$. Then the closure of $\mathfrak{M}_1$ in $(\mathfrak{M}, T)$ is clearly $\mathfrak{M} \cap F(D_1)$. Let $M \in \mathfrak{M} - \mathfrak{M}_1$. Suppose $M \supseteqq D_1$. As $M \in \mathfrak{M}$, $M \not\subseteqq M_i$ for all $M_i \in \mathfrak{M}_1$ and so, as the M_i are prime, $M^* \subseteqq M_i$ for every $M_i \in \mathfrak{M}_1$ (M^* denotes the pseudocomplement of M in the lattice of dual semi-ideals of L). Hence $M^* \subseteqq D_1 \subseteqq M$ by assumption. Consequently, $M^* = [1)$ and so M has a pseudocomplement $M^{\#}$ in $\mathfrak{D}$ and $M^{\#} = [1)$. Thus M is dense. This is a contradiction to our assumption that $M \notin \mathfrak{M}_1$. Hence $M \not\supseteqq D_1$. It follows that every member of $\mathfrak{M}$ which contains D_1 is dense and so $\mathfrak{M} \cap F(D_1) = \mathfrak{M}_1$. Thus $\mathfrak{M}_1$ is closed in $(\mathfrak{M}, T)$. As $(\mathfrak{M}, T)$ is compact it follows that $(\mathfrak{M}_1, T)$ is also compact.

For any ideal A of L, let $G(A)$ denote the set of all prime dual ideals disjoint with A. Let $G'(A) = \mathcal{P} - G(A)$. Then we have the following

Theorem 14. (i) $G(\bigvee_{i\in I} A_i) = \bigcap_{i\in I} G(A_i)$, (ii) $G(A_1 \cap A_2 \cap \cdots \cap A_n) = G(A_1) \cup G(A_2) \cup \cdots \cup G(A_n)$, (iii) $G(L) = \phi$, (iv) $G((0]) = \mathcal{P}$.

(*Here the* A_i *are ideals of* L.)

Proof. (i) It is easily seen that for any prime dual ideal B of L, $B \cap (\bigvee_{i\in I} A_i) = \phi \Leftrightarrow B \cap A_i = \phi$ for every $i \in I$. Hence (i). (ii) It is enough to prove that for a prime dual ideal B, $B \cap A_1 \cap A_2 \cap \cdots \cap A_n = \phi \Leftrightarrow B \cap A_i = \phi$ for some i $(= 1, 2, \cdots, n)$. Clearly $B \cap A_i = \phi \Rightarrow B \cap A_1 \cap A_2 \cap \cdots \cap A_n = \phi$ for every i $(= 1, 2, \cdots, n)$. Suppose $B \cap A_i \neq \phi$ for all i $(= 1, 2, \cdots, n)$. Then there exist elements $a_1, a_2, \cdots, a_n$ in $A_1, A_2, \cdots, A_n$ respectively such that $a_1, a_2, \cdots, a_n \in B$. Hence $a_1 a_2 \cdots a_n \in B$. Clearly $a_1 a_2 \cdots a_n \in A_1 \cap A_2 \cap \cdots \cap A_n$. It follows that $B \cap A_1 \cap A_2 \cap \cdots \cap A_n \neq \phi$. Consequently $B \cap A_1 \cap A_2 \cap \cdots \cap A_n = \phi \Leftrightarrow B \cap A_i = \phi$ for some i $(= 1, 2, \cdots, n)$. The proof of (ii) is now complete.

(iii) and (iv) are obvious.

The above theorem shows that G defines a closure operation in $\mathcal{P}$, thereby giving rise to a topology T' on $\mathcal{P}$.

Since $G'(A) = \mathcal{P} - G(A)$, as a consequence of Theorem 14 we have the following

Theorem 15. (i) $G'(\bigvee_{i\in I} A_i) = \bigcup_{i\in I} G'(A_i)$, (ii) $G'(A_1 \cap A_2 \cap \cdots \cap A_n) = G'(A_1) \cap G'(A_2) \cap \cdots \cap G'(A_n)$, (iii) $G'(L) = \mathcal{P}$, (iv) $G'((0]) = \phi$.

From Theorems 14 and 15 we have the following

Theorem 16. *The lattice of all open (closed) subsets of $(\mathcal{P}, T')$ is homomorphic (dually homomorphic) to the lattice of ideals of L and the mapping $A \to G'(A)$ $(A \to G(A))$ takes arbitrary lattice sums into the corresponding set unions (set intersections).*

Theorem 17. *If X is a subset of $(\mathcal{P}, T')$, then* $\mathrm{cl}\, X = G(X_0)$, *where X_0 is the product of the set complements of all the members of X.*

The proof of Theorem 17 is similar to the corresponding result of [15].

Theorem 18. *The T_1 points of $(\mathcal{P}, T)$ are identical in their totality with the anti-T_1 points of $(\mathcal{P}, T')$ and vice versa.*

Proof. Suppose A is a T_1 point of $(\mathcal{P}, T)$. Then the closure of A in $(\mathcal{P}, T)$ does not contain any other point of $(\mathcal{P}, T)$. Hence no other member of $\mathcal{P}$ contains A. In other words, A is not contained in any other member of $\mathcal{P}$. Therefore A does not belong to the closure of any other point of $(\mathcal{P}, T')$. Thus A is an anti-T_1 point of $(\mathcal{P}, T')$.

The second part is proved on similar lines.

3. **Bicomplemented lattices.** This section is devoted to a study of the additional features of the space $(\mathcal{P}, T)$ and its subspaces when the given lattice is bicomplemented. The results of this section extend the results of Balachandran [3] on Stone's topology of the distributive lattice to bicomplemented lattices.

Throughout this section, L denotes a bicomplemented lattice. Since L is closed for quasicomplements, the lattice $\mathfrak{D}$ of dual ideals of L is closed for pseudocomplements. Hence we can speak of simple dual ideals of L. The set of all simple maximal dual ideals of L is denoted by $\mathfrak{M}_2$.

Lemma 1. *In a quasicomplemented lattice L, the pseudocomplement of a dual ideal A is the product of all the prime dual ideals not containing A.*

Proof. Let $A^{\#}$ denote the pseudocomplement of A and B the product of all the prime dual ideals not containing A. It is easily seen that if a prime dual ideal contains the product of two dual ideals, then it contains at least one of them. It follows that $A^{\#} \subseteqq B$. If $A^{\#} \neq B$, then there exists $x \in B - A^{\#}$; $x \notin A^{\#}$, and so $x + y \neq 1$ for some $y \in A$. Hence by the dual of Lemma I, there exists a maximal ideal M containing $(x + y]$. Clearly $x, y \notin cM$ and so $A, B \nsubseteqq cM$. This is a contradiction to the choice of B, since it is easy to see that cM is a prime dual ideal. Hence $B = A^{\#}$.

Since L is quasicomplemented, the set of minimal prime dual ideals and the set of maximal ideals are mutually complementary. Also by Lemma 1, it follows that $[1)$ is the product of all the prime dual ideals of L. Hence as a consequence of Lemmas III, VII and VIII we have the following

Theorem 19. *The following statements concerning* L *are equivalent.*

(i) *Every prime ideal is minimal prime.*

(ii) *Every prime ideal is maximal.*

(iii) *Every prime dual ideal is minimal prime.*

(iv) *Every prime dual ideal is maximal.*

(v) $D = [1)$.

Each of the above conditions is necessary and sufficient for a bicomplemented modular lattice to be a Boolean algebra.

Theorem 20. $(\mathcal{P}, T)$ *is* Π_0 *if and only if* $D = [1)$.

Proof. In view of Theorem 7, it is enough to prove the 'only if' part.

Let $(\mathcal{P}, T)$ be Π_0. Suppose $D \neq [1)$. By Lemma III and Theorem 19 it follows that there exist prime dual ideals not containing D. Hence $F'(D) \neq \phi$. Since $(\mathcal{P}, T)$ is Π_0, $F'(D)$ contains a nonempty closed set $F(B)$. Clearly $B \neq L$ and so by Lemma I, there exists a maximal dual ideal M containing B. By Lemma II, $M \in \mathcal{P}$. Thus $M \in F(B) \subseteqq F'(D)$ and so $M \not\supseteqq D$. This is a contradiction to Lemma III. Hence $D = [1)$.

As an immediate corollary of Theorems 19 and 20 we have the following

Theorem 21. *Let* L *be a bicomplemented modular lattice. Then* $(\mathcal{P}, T)$ *is* Π_0 *if and only if* L *is a Boolean algebra.*

Theorem 22 (*cf.* [8, (3) *of Theorem* 5]). *A closed subset* $F(A)$ *of* $(\mathcal{P}, T)$ *is open if and only if* A *is simple.*

Proof. Clearly $F'(A^{\#}) \subseteqq F(A)$. Also if $F'(B) \subseteqq F(A)$ for some dual ideal B, by Lemma 1 it follows that $A \subseteqq B^{\#}$. Hence $B \subseteqq A^{\#}$ and so $F'(B) \subseteqq F'(A^{\#})$. Consequently $\operatorname{int} F(A) = F'(A^{\#})$. Hence if $F(A)$ is open, $F(A) = F'(A^{\#})$ and so $\mathcal{P} = F'(A) \cup F'(A^{\#}) = F'(A \vee A^{\#})$. By Lemma I, it follows that $A \vee A^{\#} = L$. Thus A is simple. The converse is got by retracing the steps.

Theorem 23. *The subspace* $(\mathfrak{M}_2, T)$ *is discrete.*

Proof. Clearly every member of $\mathfrak{M}_2$ is closed and hence, by Theorem 22, open in $(\mathcal{P}, T)$. Since every subset of $\mathfrak{M}_2$ is a union of members of $\mathfrak{M}_2$, it follows that every subset of $\mathfrak{M}_2$ is open. Hence the result.

Corollary 1. *A lattice with* 0, 1 *is a finite Boolean algebra if and only if* *the lattice of all dual ideals is a Boolean algebra.*

Proof. Suppose L is a finite Boolean algebra. Then clearly every dual ideal of L is principal and so $\mathfrak{D}$ is isomorphic to L. Hence $\mathfrak{D}$ is a finite Boolean algebra.

Conversely, suppose $\mathfrak{D}$ is a Boolean algebra. Then clearly L is distributive. As $\mathfrak{D}$ is a Boolean algebra, by the dual of Lemma V, it follows that every dual ideal of L is principal. Consequently $\mathfrak{D}$ and L are isomorphic and so L is a Boolean algebra. Hence $\mathcal{P} = \mathfrak{M}_2$. Thus $(\mathcal{P}, T)$ is discrete. Also, $(\mathcal{P}, T)$ is compact. Hence $\mathcal{P}$ is finite and so $\mathfrak{D}$ is finite. It follows that L is a finite Boolean algebra.

Corollary 2 (*cf.* [4, *p.* 161, *Example* 3]). *A Boolean algebra L is finite if and only if every dual ideal of L is principal.*

Proof. If every dual ideal of L is principal, $\mathfrak{D}$ is isomorphic to L and so $\mathfrak{D}$ is a Boolean algebra. Hence, by Corollary 1, L is a finite Boolean algebra.

The 'only if' part is obvious.

Theorem 24. *If X is any subset of $\mathcal{P}$ containing $\mathfrak{M}_1$, then (X, T) is compact.*

Proof. Let $Y = \mathfrak{M}_2 \cup X$. Then clearly Y is a subset of $\mathcal{P}$ containing $\mathfrak{M}$ and so by Theorem 12, (Y, T) is compact. By Theorem 23, every point of $\mathfrak{M}_2$ is open in $(\mathcal{P}, T)$. Now $Y - X \subseteqq \mathfrak{M}_2$, so that every point of $Y - X$ is open in $(\mathcal{P}, T)$ and hence open in (Y, T). Consequently $Y - X$ is open in (Y, T) and so X is closed in (Y, T). Since (Y, T) is compact, it follows that (X, T) is compact.

Let $\mathfrak{N}$ and $\mathfrak{N}_1$ denote respectively the set of minimal prime dual ideals and the set of normal prime dual ideals.

Theorem 25. *The subspace $(\mathfrak{N}_1, T)$ is discrete.*

Proof. Let $X = \{N_i : i \in I\}$ be any subset of $\mathfrak{N}_1$ and $N \in \mathrm{cl}\, X$. Then $N \supseteqq \bigcap_{i \in I} N_i$. Suppose $N \not\supseteqq N_i$ for all $i \in I$. Then for each $i \in I$, $N \supseteqq N_i^{\#}$ and so $N \supseteqq \bigvee_{i \in I} N_i^{\#}$. It follows that $N \supseteqq (\bigcap_{i \in I} N_i) \vee (\bigvee_{i \in I} N_i^{\#})$. Consequently

$$N^{\#} \subseteqq (\bigcap_{i \in I} N_i)^{\#} \cap (\bigvee_{i \in I} N_i^{\#})^{\#} = (\bigcap_{i \in I} N_i)^{\#} \cap (\bigcap_{i \in I} N_i) = [1).$$

This is not possible since N is normal. Hence $N \supseteqq N_j$ for some $j \in I$. Since L is quasicomplemented, it is easily seen that every normal prime dual ideal is minimal prime. (This can be proved by using the analogue of Lemma IV for dual ideals.) It follows that $N = N_j$. Thus $N \in X$ and so $\mathrm{cl}\, X = X$. Hence $(\mathfrak{N}_1, T)$ is discrete.

Theorem 26. *The subspace $(\mathfrak{N}, T)$ is T_3.*

Proof. Since no minimal prime dual ideal contains any other minimal prime dual ideal $(\mathfrak{N}, T)$ is T_1.

Let X be any nonvoid closed subset of $\mathfrak{N}$ and $A \notin X$ $(A \in \mathfrak{N})$. Then $X = \mathfrak{N} \cap F(B)$ for some dual ideal B. Clearly $A \not\supseteq B$. Hence there exists $b \in B$ such that $b \notin A$. By the analogue of Lemma VI, $\overline{b} \in A$. ($\overline{b}$ is the quasicomplement of b.) Hence $A \in F([\overline{b}))$. Set $X_1 = \mathfrak{N} \cap F([b))$ and $X_2 = \mathfrak{N} \cap F([\overline{b}))$. Then clearly $X_1 \supseteq X$, $A \notin X_1$ and $A \in X_2$. Now $X \cap X_2 = \mathfrak{N} \cap F(B) \cap F([\overline{b})) \subseteq \mathfrak{N} \cap (F([b)) \cap F([\overline{b}))) = \phi$ (by the dual of Lemma VI). $X_1 \cup X_2 = \mathfrak{N} \cap (F([b)) \cup F([\overline{b}))) = \mathfrak{N}$.

From the above it follows that $(\mathfrak{N}, T)$ is regular. This completes the proof.

Lemma 2. *The maximal (minimal prime) dual ideals of* L *are identical in their totality with the* T_1 *(anti-*T_1*) points of* $(\mathfrak{P}, T)$.

The proof of Lemma 2 is straightforward and follows from Theorem 3.

Theorem 27. T *is stronger (weaker) than* T' *at a point* A *of* $\mathfrak{P}$ *if and only if* A *is a* T_1 *(anti-*T_1*) point of* $(\mathfrak{P}, T)$.

Proof. If A is a T_1 point of $(\mathfrak{P}, T)$, by Lemma 2, A is a maximal dual ideal of L. Let $F'(B)$ be any T-neighbourhood of A. Then $A \not\supseteq B$, so that there exists $b \in L$ such that $b \in B$, $b \notin A$. As A is maximal, $A \vee [b) = L$. Hence there exists $a \in A$ with $ab = 0$. Clearly $b^* \geqq a$ and so $b^* \in A$. Thus $(b^*]$ meets A and consequently $A \in G'((b^*])$. Also, if a prime dual ideal meets $(b^*]$, then it contains b^* and so it cannot contain B, as $b \in B$. Hence $G'((b^*]) \subseteq F'(B)$. Thus any T-neighbourhood of A contains a T'-neighbourhood of A. Hence T is stronger than T' at A.

Let T be stronger than T' at A. Suppose A is not a T_1 point of $(\mathfrak{P}, T)$. Then, by Lemma 2, A is not a maximal dual ideal and so by Lemma I, there exists a maximal dual ideal M such that M properly contains A. Clearly $A \in F'(M)$, so that, by hypothesis there exists a T'-neighbourhood $G'(B)$ of A such that $G'(B) \subseteq F'(M)$. Clearly $F(M) \subseteq G(B)$ and $M \in F(M)$. Hence $M \cap B = \phi$ and so $A \cap B = \phi$. This is a contradiction to the fact that $A \in G'(B)$. Hence A is a T_1 point of $(\mathfrak{P}, T)$.

Now suppose A is an anti-T_1 point of $(\mathfrak{P}, T)$. Then, by Lemma 2, A is a minimal prime dual ideal of L. Let $G'(B)$ be a T'-neighbourhood of A. Then $A \cap B \neq \phi$. Let $x \in A \cap B$. Then, as $x \in A$ and A is minimal prime, $A \not\supseteq [\overline{x})$ and so $A \in F'([\overline{x}))$ ($\overline{x}$ denotes the quasicomplement of x in L). Also, any prime dual ideal not containing $[\overline{x})$ meets B, as $x \in B$. Hence $F'([\overline{x})) \subseteq G'(B)$. Thus any T'-neighbourhood of A contains a T-neighbourhood of A. Hence T is weaker than T' at A.

Let T be weaker than T' at A. Suppose A is not an anti-T_1 point of $(\mathfrak{P}, T)$. Then, by Lemma 2, A is not a minimal prime dual ideal of L and so, by

Lemma IV, there exists a minimal prime dual ideal M such that M is properly contained in A. The set complement cM of M, which is a maximal ideal, meets A. Hence $A \in G'(cM)$. Thus $G'(cM)$ is a T'-neighbourhood of A and so, by hypothesis, there exists a T-neighbourhood $F'(B)$ of A such that $F'(B) \subseteqq G'(cM)$. Clearly $G(cM) \subseteqq F(B)$ and $M \in G(cM)$. Hence $M \supseteqq B$ and so A properly contains B. This is a contradiction to the fact that $A \in F'(B)$. Hence A is an anti-T_1 point of $(\mathcal{P}, T)$.

As a consequence of Theorems 27 and 19 we have the following

Theorem 28. *Let L be a bicomplemented modular lattice. Then $(\mathcal{P}, T)$ and $(\mathcal{P}, T')$ are homeomorphic if and only if L is a Boolean algebra.*

4. **Ideals of the quotient lattice.** We have proved in [11] that if L is a pseudocomplemented lattice, the relation defined by $a \equiv b \Leftrightarrow a^* = b^*$ is a congruence relation in L and the quotient lattice $\hat{L}$ is a Boolean algebra. In this section we shall obtain some results concerning the relations between the ideals of L and those of $\hat{L}$; these results are applied to study the relations between Stone's topologies of prime dual ideals of L and $\hat{L}$.

It is easily seen that if $\hat{a}$ is the congruence class determined by a, then the mapping $f: a \longrightarrow \hat{a}$ is a homomorphism of L onto $\hat{L}$.

Theorem 29. *$f(A)$ is an ideal or a dual ideal of $\hat{L}$ according as A is an ideal or a dual ideal of L.*

Proof. Let $\hat{a}, \hat{b} \in f(A)$ $(a, b \in A)$, A being an ideal of L. Then $\hat{a} + \hat{b} = \widehat{(a + b)} \in f(A)$, as $a + b \in A$. Let $\hat{c} \leqq \hat{a}$. Then $\hat{c} = \hat{c}\hat{a} = \widehat{ca} \in f(A)$, as $ca \in A$. Thus $f(A)$ is an ideal of $\hat{L}$.

The result for dual ideals is proved on similar lines.

Theorem 30. *Let $\hat{A}$ be an ideal of $\hat{L}$ and $A = f^{-1}(\hat{A})$. Then A is an ideal of L and it is the greatest of all the ideals X of L for which $f(X) = \hat{A}$. Similar result holds for dual ideals.*

Proof. Let $a, b \in A$. Then $\hat{a}, \hat{b} \in \hat{A}$ and so $\widehat{(a + b)} = \hat{a} + \hat{b} \in \hat{A}$. Hence $a + b \in A$. Let $c \leqq a$. Then $c = ca$, so that $\hat{c} = \widehat{ca} = \hat{c}\hat{a} \in \hat{A}$. Hence $c \in A$. Thus A is an ideal of L. Now let X be any ideal of L with $f(X) = \hat{A}$ and $x \in X$. Then $\hat{x} \in \hat{A}$ and so $x \in A$. It follows that $X \subseteqq A$.

The second part is proved on similar lines.

Theorem 31. *There is a 1-1 reversible correspondence between the set of maximal dual ideals of L and the set of prime dual ideals of $\hat{L}$.*

Proof. Let M be a maximal dual ideal of L and $\hat{M} = f(M)$. By Theorem 29, $\hat{M}$ is a dual ideal of $\hat{L}$. $\hat{0} \notin \hat{M}$. (For, otherwise for some $a \in M$, $\hat{a} = \hat{0}$ and so

$a^* = 0^* = 1$. Consequently $a = 0$ which is absurd since $M \neq L$.) Hence $\hat{M} \neq \hat{L}$. Let $\hat{A} \supseteqq \hat{M}$, $\hat{A}\,(\neq \hat{L})$ being a dual ideal of $\hat{L}$. Clearly $f^{-1}(\hat{A}) \supseteqq f^{-1}(\hat{M}) = M$ by Theorem 30. But, as $\hat{A} \neq \hat{L}$, $f^{-1}(\hat{A}) \neq L$. Since M is maximal, it follows that $f^{-1}(\hat{A}) = M$. Hence $\hat{A} = ff^{-1}(\hat{A}) = f(M) = \hat{M}$. Thus $\hat{M}$ is maximal and so, by Lemma II, it is prime.

Now suppose $\hat{M}$ is a prime dual ideal of $\hat{L}$. As $\hat{L}$ is a Boolean algebra, by Theorem 19, $\hat{M}$ is maximal. Let $M = f^{-1}(\hat{M})$. Then by Theorem 30, M is a dual ideal of L. We shall show that M is maximal. If $A \supseteqq M$, $A\,(\neq L)$ being a dual ideal of L, $f(A) \supseteqq f(M) = \hat{M}$. By Theorem 29, $f(A)$ is a dual ideal of $\hat{L}$. Also $\hat{0} \notin f(A)$; for otherwise for some $a \in A$, $a^* = 0^* = 1$ and so $a = 0$ which is a contradiction to our assumption that $A \neq L$. Hence $f(A) = \hat{M}$. It follows that $A = f^{-1}f(A) = f^{-1}(\hat{M}) = M$. Thus M is maximal.

If M_1 is any maximal dual ideal of L with $f(M_1) = \hat{M}$, then $M_1 = f^{-1}(\hat{M}) = M$. Thus the mapping f induces a 1-1 reversible correspondence between maximal dual ideals of L and prime dual ideals of $\hat{L}$.

Since maximal dual ideals and minimal prime ideals of L are complementary, as are the prime ideals and prime dual ideals of $\hat{L}$, as an immediate consequence of Theorem 31 we have the following

Theorem 32. *There is a* 1-1 *reversible correspondence between the set of minimal prime ideals of* L *and the set of prime ideals of* $\hat{L}$.

Theorem 33. $(\mathfrak{M}, T)$ *is homeomorphic to* $(\hat{\mathcal{P}}, \hat{T})$, *where* $(\hat{\mathcal{P}}, \hat{T})$ *is Stone's space of prime dual ideals of* $\hat{L}$.

Proof. By Theorem 31, the map $g : M \to \hat{M}$ is a 1-1 map of $\mathfrak{M}$ onto $\hat{\mathcal{P}}$. We shall prove that g is a homeomorphism. Let X be any closed subset of $(\mathfrak{M}, T)$. Then $X = \mathfrak{M} \cap F(A)$, A being a dual ideal of L. Let $\hat{A} = \bigcap \hat{M}_i$, the M_i being maximal dual ideals of L containing A. Then

$$g(\mathfrak{M} \cap F(A)) = g(\mathfrak{M}) \cap g(F(A)) = \hat{\mathcal{P}} \cap \hat{F}(\hat{A}) = \hat{F}(\hat{A})$$

where $\hat{F}(\hat{A})$ denotes the set of prime dual ideals of $\hat{L}$ containing $\hat{A}$. Thus g takes closed subsets of $(\mathfrak{M}, T)$ to closed subsets of $(\hat{\mathcal{P}}, \hat{T})$ and is therefore a closed map.

Now let $\hat{X} = \hat{F}(\hat{A})$ be any closed subset of $(\hat{\mathcal{P}}, \hat{T})$. Then $g^{-1}(\hat{X}) = g^{-1}(\hat{F}(\hat{A})) = \mathfrak{M} \cap F(A)$, where $A = \bigcap g^{-1}(\hat{M}_i)$, $\hat{M}_i$ being prime factors of $\hat{A}$. Thus g^{-1} takes closed subsets of $(\hat{\mathcal{P}}, \hat{T})$ to closed subsets of $(\mathfrak{M}, T)$ and is therefore a closed map. It follows that g is a homeomorphism of $(\mathfrak{M}, T)$ onto $(\hat{\mathcal{P}}, \hat{T})$.

Theorem 34. $(\mathfrak{M}, T)$ *is normal.*

Proof. Since $\hat{L}$ is a Boolean algebra, by Theorem 19, every dual ideal of $\hat{L}$

is minimal prime. Hence, by Theorem 26, $(\hat{\mathcal{P}}, \hat{T})$ is T_3 and hence Hausdorff. Also $(\hat{\mathcal{P}}, \hat{T})$ is compact. Hence $(\hat{\mathcal{P}}, \hat{T})$ is normal. Now the result follows from Theorem 33.

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